SOME PROOFS

IN

ELEMENTARY GEOMETRY

BY

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JONES' FIVE-PLACE LOGARITHMS.

JONES' LOGARITHMIC TABLES.

FOUR-PLACE, SIX-PLACE, AND TEN-PLACE.

(SEE THE THIRD COVER PAGE.)

SOME PROOFS IN ELEMENTARY GEOMETRY.

The proofs here given are such proofs as I have been accustomed to give in my own classes. They are written out for my pupils; but if any teacher finds them useful I shall be very glad. They are alternative proofs that may be used instead of proofs by limits. They are meant to be rigorous.
G. W. J.

In this paper the sign $\because$ stands for *because*, $\therefore$ for *therefore*, $\neq$ for *not equal to*, $\doteq$ for *approaches*, meaning thereby that if one magnitude approaches another magnitude of the same kind, they come at last to differ by less than any assigned magnitude, and tend toward equality.

By a *lemma* is meant a theorem that is assumed to have been proved, and which is needed in the proof of the theorem under consideration.

By a *ratio* is meant that number by which the consequent of the ratio is to be multiplied to give the antecedent.

By the *product* of *two lines* is meant the rectangle of which the two lines are adjacent sides. If they be broken or curved, they are to be straightened first.

By the *product* of *a line and a surface* is meant the rectangular parallelopiped whose altitude is the given line, and whose base is a rectangle equal in area to the given surface. If this surface be broken or curved, it is to be flattened first.

By a *line* is meant a straight line. If a line be broken or curved, it is so named.

By *two equal lines* is meant two lines that are equal in length.

By *two equal surfaces* is meant two surfaces that are equal in area.

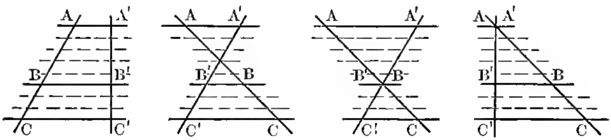
By *two equal solids* is meant two solids that are equal in volume.

THEOR. 1. *If three parallel lines cut two other lines, the like segments of the two lines are proportional.*

LEMMA. If three or more parallel lines cut two other lines, and if one of these lines be cut into equal parts, so is the other.

Let one of the lines be cut by the parallels in A, B, C, and the other in A′, B′, C′; then will AB/BC = A′B′/B′C′.

(1) AB, BC *commensurable.*



Let the common measure of AB, BC go into AB m times and into BC n times, and through the points of division draw lines parallel to the given parallels;

then ∵ AB is cut into m equal parts, so is A′B′, [lem.

and ∵ BC is cut into n equal parts, so is B′C′,

∴ AB is got from BC by dividing BC into n equal parts and taking m of these parts,

and A′B′ is got from B′C′ in the same way;

i.e. AB has the same relation to BC as A′B′ has to B′C′,

and AB/BC = A′B′/B′C′. Q. E. D.

e.g. if $m = 5$ and $n = 3$;

then ∵ AB is got from BC by multiplying BC by $\frac{5}{3}$,

and A′B′ is got from B′C′ by multiplying B′C′ by $\frac{5}{3}$,

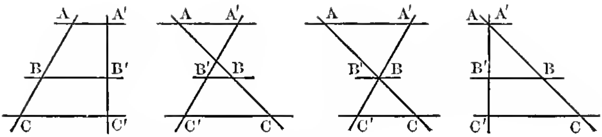
∴ AB/BC = A′B′/B′C′. Q. E. D.

(2) AB, BC *not commensurable.*

Let BC be cut into n equal parts, and let AB contain more than m of these parts and less than $m+1$ of them.

Draw lines through these points of division parallel to the given parallels;

then ∵ B′C′ is thus cut into n equal parts and A′B′ contains more than m of these parts and less than $m+1$ of them, [lem.



$\therefore m/n < \text{AB}/\text{BC} < (m+1)/n$

and $\quad m/n < \text{A}'\text{B}'/\text{B}'\text{C}' < (m+1)/n$;

$\therefore$ there are four ratios whereof m/n is least, $(m+1)/n$ is greatest, and the constant ratios AB/BC and A′B′/B′C′ lie between them.

If possible let $\text{AB}/\text{BC} \neq \text{A}'\text{B}'/\text{B}'\text{C}'$;

then $\because$ whatever the difference of the constant ratios AB/BC and A′B′/B′C′, that difference is constant,

and $\because$ the ratios m/n and $(m+1)/n$ are variables such that their difference $1/n$ approaches 0 when n is made very great,

$\therefore$ the least of the four ratios can be made to differ from the greatest of them by less than the two intermediate ratios differ ; which is absurd ;

$\therefore$ the ratios AB/BC and A′B′/B′C′ do not differ, and they are equal. Q. E. D.

e.g. if BC be the side of a square and AB its diagonal ;

then $\because \text{AB}/\text{BC} = \sqrt{2} = 1.4142, \ldots$

and $\quad 1 < \sqrt{2} < 2,\ 14/10 < \sqrt{2} < 15/10, \ldots$

$14142/10000 < \sqrt{2} < 14143/10000, \ldots$

$m/n < \sqrt{2} < (m+1)/n$, wherein n is some very great number, and m is still greater ;

$\therefore 1/n$, the difference of the variable ratios m/n and $(m+1)/n$, can be made smaller than any given fraction when n is made great enough.

If, then, the difference of the constant ratios AB/BC and A′B′/B′C′ be a millionth or any other fixed fraction, the difference of the ratios m/n and $(m+1)/n$ can be made smaller than this fraction, say a billionth; and the absurdity is apparent.

THEOR. 2. *Two rectangles that have equal bases are proportional to their altitudes.*

LEMMA. Two rectangles that have equal bases and the same altitude are equal.

THEOR. 3. *Two angles at the center of a circle are proportional to their subtending arcs.*

LEMMA. If two angles at the center of a circle be equal, their subtending arcs are equal.

NOTE.—The proofs of theors. 2, 3 are like that of theor. 1.

THEOR. 4. *If there be two similar regular polygons, the one inscribed in a circle and the other circumscribed about it, and if the number of the sides be doubled again and again, so that the polygons remain regular and similar, then* (1) *the difference of their perimeters may be made less than any assigned line and that of their surfaces less than any assigned surface;* (2) *the surface of a circle is half the product of the perimeter of the circle and its radius.*

LEM 1. If two polygons be similar, their perimeters are proportional to any two like lines, and their surfaces are proportional to the squares of such lines.

LEM. 2. The surface of a regular polygon is half the product of its perimeter and apothem.

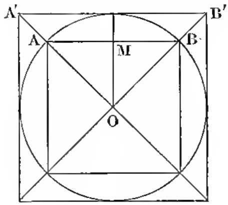
LEM. 3. If there be two polygons, the one inscribed in a circle and the other circumscribed about it, the perimeter of the circle is greater than that of the inner polygon and less than that of the outer polygon; and so of their surfaces.

LEM. 4. The difference of two sides of a triangle is less than the third side.

Let OA be a circle, r its radius, c its perimeter, C its surface.

Let AB be the side of an inscribed square, and A′B′ that of a circumscribed square; and let the number of sides of both polygons be doubled again and again, so that the polygons remain regular and similar, and the sides grow shorter and shorter.

Let a be the apothem of the inner polygon, p its perimeter, and s its surface; and let P be the perimeter of the outer polygon and S its surface.



then $\because$ $\mathrm{P}/p=r/a$, [lem. 1.

$\therefore$ $(\mathrm{P}-p)/(r-a)=\mathrm{P}/r$; [prop'n.

and $\because$ $r-a<$ half of one of the short sides of the inner polygon, and this side approaches 0, [lem. 4.

$\therefore$ $r-a \doteq 0$;

and $\because$ $\mathrm{P}<8r$ and $\mathrm{P}-p<8(r-a)$,

$\therefore$ $\mathrm{P}-p \doteq 0$. Q. E. D.

So $\because$ $\mathrm{S}/s=r^2/a^2$, [lem. 1.

$\therefore$ $(\mathrm{S}-s)/(r^2-a^2)=\mathrm{S}/r^2$;

and $\because$ $\mathrm{S}<4r^2$ and $r^2-a^2=(r+a)(r-a)\doteq 0$,

$\therefore$ $\mathrm{S}-s<4(r^2-a^2)\doteq 0$. Q. E. D.

So $\because$ $a<r$ and $p<c<\mathrm{P}$, [lem. 3.

$\therefore$ $\frac{1}{2}pa<\frac{1}{2}cr<\frac{1}{2}\mathrm{P}r$; [ax. ineq.

and $\because$ $s<\mathrm{C}<\mathrm{S}$ and $s=\frac{1}{2}pa$, $\mathrm{S}=\frac{1}{2}\mathrm{P}r$, [lem. 3, 2.

$\therefore$ there are four surfaces whereof s and its equal $\frac{1}{2}pa$ is least; S and its equal $\frac{1}{2}\mathrm{P}r$ is greatest; and the constant surfaces C and $\frac{1}{2}cr$ lie between them.

If possible let $\mathrm{C} \neq \frac{1}{2}cr$;

then $\because$ whatever the difference of these two constants, that difference is constant, and the two variables S, s can be made to differ by less, [above.

$\therefore$ the least of the four surfaces can be made to differ from the greatest of them by less than the two intermediate surfaces differ; which is absurd;

$\therefore$ these two surfaces do not differ, and they are equal.

THEOR. 5. *Two rectangular parallelopipeds that have equal bases are proportional to their altitudes.*

LEMMA. Two rectangular parallelopipeds that have equal bases and the same altitude are equal.

THEOR. 6. *Two diedral angles are proportional to their plane angles.*

LEMMA. If two diedral angles be equal, their plane angles are equal.

THEOR. 7. *Two lunes on the same sphere are proportional to their angles; and so are two such spherical wedges.*

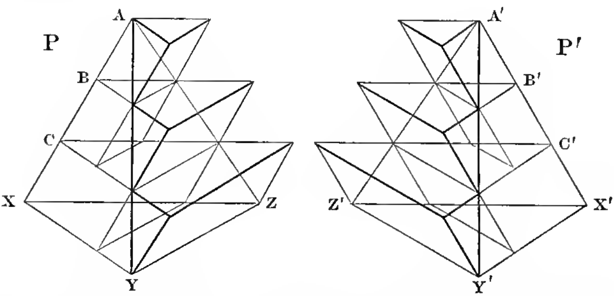
LEMMA. If two lunes on the same sphere be equal, their angles are equal; and so if two such spherical wedges be equal.

NOTE.—The proofs of theors. 5, 6, 7 are like that of theor. 1.

THEOR. 8. *If two triangular pyramids have the same altitude and bases that are equal in area, the pyramids are equal in volume.*

LEM. 1. If two pyramids have the same altitude and bases that are equal in area, sections made by planes parallel to the bases and at equal distances from the vertices are equal in area.

LEM. 2. If two triangular prisms have the same altitude and bases that are equal in area, the prisms are equal in volume.



Let the two triangular pyramids P, P′ have the same altitude and their bases XYZ, X′Y′Z′ be equal in area; then are P, P′ equal in volume.

For, cut the edges AX, A′X′ into the same number of equal parts at B, C, . . . B′, C′ . . . and through these points of division draw planes parallel to the bases;

then ∵ the sections at B, B′ are equidistant from the vertices A, A′,

∴ they are equal in area; [lem. 1.

and so are the sections at C, C′ . . .

On the base XYZ, and on the sections at B, C, . . . construct prisms having one edge on AX and reaching each to the section above it.

Call these the *outer prisms* of P.

Below the sections as upper bases construct like prisms, and call them the *inner prisms* of P.

So construct the sets of outer, and of inner, prisms of P′;

then ∵ AX, A′X′ are cut into the same number of equal parts,

∴ all the prisms have the same altitude;

and ∵ the two outer prisms on XYZ, X′Y′Z′ have the same altitude and bases that are equal in area,

∴ these two prisms are equal in volume, [lem. 2.
and so are the others, two and two, to the top of both pyramids,

∴ the set of outer prisms of P is equal in volume to the set of outer prisms of P′;

and so are the two sets of inner prisms equal in volume.

And ∵ the first outer prism of P is equal to the first inner prism, counting from the top, the second to the second, and so on, [lem. 2.

∴ the sum of the outer prisms of P exceeds the sum of the inner prisms by the outer prism at the bottom;

and so for the outer and inner prisms of P′.

Let the edges AX, A′X′ be cut into more and more equal parts, so that the plates (prisms) are of equal thickness but growing thinner and thinner.

Let V, V' stand for the sets of outer plates of P, P', and v, v' for the sets of inner plates;

then $\because V = V'$ and $v = v'$, [above.

and V, v are variables such that $V - v \doteq 0$, and so are V', v', [above.

and $\because v < P < V$ and $v' < P' < V'$,

$\therefore$ there are four solids whereof v and its equal v' is least; V and its equal V' is greatest; and the constants, the pyramids P, P', lie between them.

If possible let $P \neq P'$;

then $\because$ whatever the difference of these two constants, that difference is constant, and the two variables V, v, can be made to differ by less,

$\therefore$ the least of the four solids can be made to differ from the greatest of them by less than the two intermediate solids differ; which is absurd;

$\therefore$ the two pyramids do not differ, and they are equal.

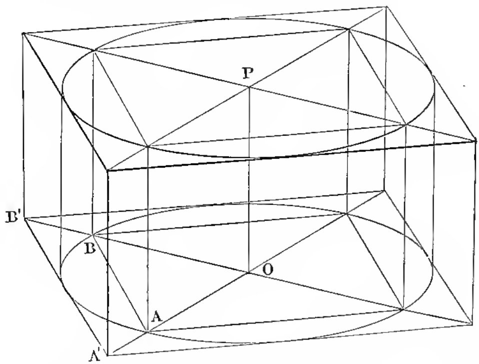
NOTE.—It is to be observed that the inequalities $v < P < V$ and $v' < P' < V'$ above are independent of each other, and are in no manner derived one from the other. This independence holds with all the pairs of inequalities that occur in the later proofs, and should be emphasized in giving the proofs.

THEOR. 9. *The lateral surface of a cylinder of revolution is the product of its altitude and the perimeter of its base; and its volume is the product of its altitude and the surface of its base.*

LEMMA. The lateral surface of a right prism is the product of its altitude and the perimeter of its base; and its volume is the product of its altitude and the surface of its base.

Let OP–A be a cylinder of revolution, OP–$AB\ldots$ a regular inscribed prism, and OP–$A'B'\ldots$ a regular circumscribed prism having the same number of faces, and let the number of faces be doubled again and again, so that the bases remain regular and similar polygons and the faces of the prisms become very narrow.

Let p be the perimeter of the inner base, P that of the outer base, c that of the circle; let s be the lateral surface of the inner prism, S that of the outer prism, *cyl.* that of the cylinder; and let h be the altitude of the cylinder and of the prisms;



then ∵ P, p are variables such that $\mathrm{P}-p\doteq 0$, [theor. 4.

∴ $h\mathrm{P}$, hp are variables such that $h\mathrm{P}-hp\doteq 0$;

and ∵ $s=hp$, $\mathrm{S}=h\mathrm{P}$, and $p<c<\mathrm{P}$, [lem.

and $hp<hc<h\mathrm{P}$ and $s<cyl.<\mathrm{S}$, [ax. ineq.

∴ there are four surfaces whereof s and its equal hp is least; S and its equal $h\mathrm{P}$ is greatest; and the constants *cyl.* and hc lie between them.

If possible let $cyl. \neq hc$;

then ∵ whatever the difference of these two constants, that difference is constant, and the two variables, hp, $h\mathrm{P}$ can be made to differ by less, [theor. 4.

∴ the least of the four surfaces can be made to differ from the greatest of them by less than the two intermediate surfaces differ; which is absurd;

∴ the surface of the cylinder does not differ from the product of the altitude and the perimeter of the base, and they are equal. Q. E. D.

So let b be the area of the inner base, B that of the outer base, C that of the circle; and let v be the volume of the inner prism, V that of the outer prism, *cyl.* that of the cylinder;

then $\because$ B, b are variables such that $B - b \doteq 0$, [theor. 4.

$\therefore$ hB, hb are variables such that $hB - hb \doteq 0$;

and $\because$ $v = hb$ $V = hB$, and $b < C < B$, [lem. 1.

and $hb < hC < hB$ and $v < cyl. < V$,

$\therefore$ there are four solids whereof v and its equal hb is least; V and its equal hB is greatest; and the constants *cyl.* and hC lie between them.

If possible let $cyl. \neq hC$;

then $\because$ whatever the difference of these two constants, that difference is constant, and the two variables hb, hB can be made to differ by less,

$\therefore$ the least of the four solids can be made to differ from the greatest of them by less than the two intermediate solids differ; which is absurd;

$\therefore$ the volume of the cylinder does not differ from the product of the altitude and the area of the base, and they are equal.

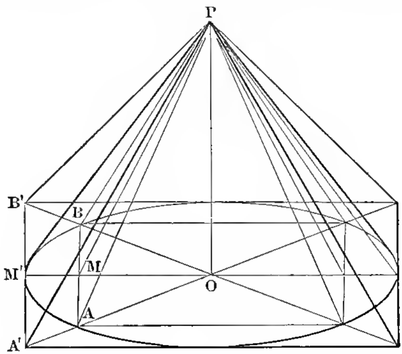
NOTE.—For the lateral surface of a cylinder whose right section is a circle, the proof is substantially the same as for the surface of a cylinder of revolution; and so is that for the volume of an oblique cylinder with circle base.

THEOR. 10. *The lateral surface of a cone of revolution is half the product of its slant height and the perimeter of its base; and the volume of such a cone is a third of the product of its altitude and the area of its base.*

LEM. 1. The lateral surface of a regular pyramid is half the product of its slant height and the perimeter of its base; and the volume of such a pyramid is a third part of the product of its altitude and the area of its base.

LEM. 2. If the lines a, b be so related to the lines a', b' that in length a approaches a', and b approaches b', then, in area, the rectangle $a \cdot b$ approaches the rectangle $a' \cdot b'$.

LEM. 3. If a perpendicular and two oblique lines be drawn from a point to a plane, that line which meets the plane further from the foot of the perpendicular is the longer.



Let OP–A be a cone of revolution, OP–AB ... a regular inscribed pyramid, and OP–A′B′ ... a regular circumscribed pyramid having the same number of faces, and let the number of the sides of the bases be doubled again and again, so that the bases remain regular and similar polygons and the faces of the pyramid become very narrow.

Let p be the perimeter of the inner base, P that of the outer base, c that of the circle; let s be the lateral surface of the inner pyramid, S that of the outer pyramid, *cone* that of the cone; and let l be the slant height of the inner pyramid, L that of the outer pyramid and the cone;

then ∵ L is constant and P, p, l, are variable lines such that $\text{P}-p \doteq 0$ and $\text{L}-l \doteq 0$, [theor. 4, lem. 3.

∴ LP, lp are variable rectangles such that $\text{LP}-lp \doteq 0$;

and ∵ $l<\text{L}$ and $p<c<\text{P}$, [lem. 3, theor. 4.

∴ $\frac{1}{2}lp<\frac{1}{2}\text{L}c<\frac{1}{2}\text{LP}$; [ax. ineq.

and ∵ $s<cone<\text{S}$,

∴ there are four surfaces whereof s and its equal $\frac{1}{2}lp$ is least; S and its equal $\frac{1}{2}\text{LP}$ is greatest; and the constants *cone* and $\frac{1}{2}\text{L}c$ lie between them.

If possible let $cone \neq \frac{1}{2}\text{L}c$;

then $\because$ whatever the difference of these two constants, that difference is constant, and the two variables $\frac{1}{2}lp$ and $\frac{1}{2}\text{LP}$ can be made to differ by less,

$\therefore$ the least of the four surfaces can be made to differ from the greatest of them by less than the two intermediate surfaces differ; which is absurd;

$\therefore$ the surface of the cone does not differ from half the product of its slant height and the perimeter of its base; and they are equal. Q. E. D.

NOTE.—For the volume of the cone, the proof is like that for the volume of the cylinder.

THEOR. 11. *The surface of a frustum of a cone of revolution is half the product of its slant height and the sum of the perimeters of its bases; and the volume of such a frustum is a third of the product of its altitude and the sum of its bases and a mean proportional between them.*

LEM. 1. If two cones of revolution be similar, the perimeters of their bases are proportional to their slant heights; and the areas of their bases to the squares of their altitudes.

LEM. 2. If three magnitudes of the same kind, a, b, c, be so related that the ratios a/b, b/c are equal, then is the ratio a/c the square of the ratio a/b and of the ratio b/c.

Let L be the slant height of the cone, l that of the top cut off, $\text{L}-l$ that of the frustum; and let P be the perimeter of the base of the cone, p that of the top;

then is $\frac{1}{2}(\text{L}-l)(\text{P}+p)$ the surface of the frustum.

For $\because$ $\text{P}/p=\text{L}/l$, [lem. 1.

$\therefore$ $l\text{P}=\text{L}p$,

and $\frac{1}{2}(\text{L}-l)(\text{P}+p)=\frac{1}{2}(\text{LP}+\text{L}p-l\text{P}-lp)=\frac{1}{2}(\text{LP}-lp)$;

and $\because$ $\frac{1}{2}\text{LP}$ is the surface of the whole cone and $\frac{1}{2}lp$ that of the top,

$\therefore$ $\frac{1}{2}(\text{LP}-lp)$ is the surface of the frustum,

and $\frac{1}{2}(\text{L}-l)(\text{P}+p)=\frac{1}{2}(\text{LP}-lp)$, the surface sought.

Q. E. D.

So let H be the altitude of the cone, h that of the top, $H-h$ that of the frustum, B the base of the cone, b that of the top, and M a mean proportional between B and b;

then is $\frac{1}{3}(H-h)(B+b+M)$ the volume of the frustum.

For $\because B/b=H^2/h^2$, [lem. 1.

and $B/M=M/b$, [def. mean propor'l.

$\therefore B/M=H/h$ and $M/b=H/h$, [lem. 2.

$\therefore hB=HM$ and $hM=Hb$;

and $\because \frac{1}{3}(H-h)(B+b+M)$

$$=\tfrac{1}{3}(HB+Hb+HM-hB-hb-hM)=\tfrac{1}{3}(HB-hb),$$

and $\frac{1}{3}HB$ is the volume of the cone and $\frac{1}{3}hb$ that of the top,

$\therefore \frac{1}{3}(HB-hb)$ is the volume of the frustum,

and $\frac{1}{3}(H-h)(B+b+M)=\frac{1}{3}(HB-hb)$, the volume sought.

NOTE 1.—The statement and proof of this theorem apply without change to the frustum of a pyramid.

NOTE 2.—So far as concerns the volume of the frustum, the statement and proof of the theorem apply to any cone with circle base.

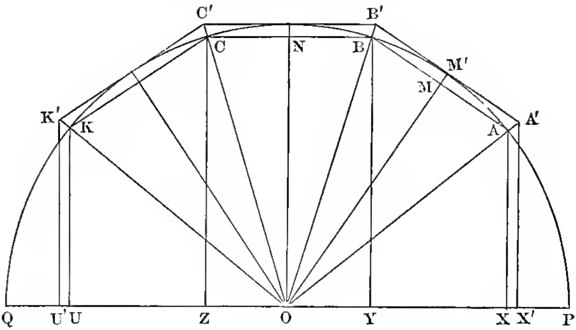
THEOR. 12. *The surface of a zone is the product of its altitude and the perimeter of a great circle of the sphere; and the volume of the spherical sector, whose base is this zone, is a third of the product of the surface of the zone and a radius of the sphere.*

LEM. 1. If an isosceles triangle be revolved about a line lying in its plane and passing through its vertex but not crossing the triangle, the surface generated by the base is the product of the projection of the base on the axis and the perimeter of a circle whose radius is the perpendicular from the vertex to the base; and the volume of the ring generated by the triangle is a third of the product of this perpendicular and this surface.

LEM. 2. If an arc of a circle, its chord, and a broken line made up of the tangents at its ends, be revolved about a diameter of the circle that does not cut the arc, the surface

generated by the arc is greater than that generated by the chord and less than that generated by the broken line.

LEM. 3. If the line a be so related to the line a' that in length a approaches a', and the rectangle $b \cdot c$ be so related to the rectangle $b' \cdot c'$ that in area $b \cdot c$ approaches $b' \cdot c'$, then in volume the rectangular parallelopiped $a \cdot b \cdot c$ approaches the rectangular parallelopiped $a' \cdot b' \cdot c'$.



Let O be the center of a circle, OA its radius, AK an arc, PQ a diameter not cutting AK.

Divide AK into a great number of equal arcs AB, BC . . .; draw the chords AB, BC . . .; and draw A'B', B'C' . . . parallel to AB, BC . . ., tangent to the arc AK, and cut off by the radii OA, OB, OC . . . ;

then the isosceles triangles OAB, OBC . . . are all equal, and so are OA'B', OB'C' . . .

Draw the perpendiculars AX, BY, CZ . . . KU, A'X', K'U' to PQ, and revolve the whole figure about PQ;

then three surfaces of revolution are generated:
the zone by the arc AK,
the inner surface by the broken line of chords,
the outer surface by the broken line of tangents;

and three solids of revolution:
the spherical sector bounded by the zone and the two cones generated by the radii OA, OK,

the inner solid bounded by the inner surface and the cones,

the outer solid bounded by the outer surface and the cones.

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(1) *The surface of the zone* AK *is the product of its altitude* XU *and the perimeter of a great circle.*

For ∵ surf. AB = XY · per. OM, [lem. 1.

surf. BC = YZ · per. OM,

. . . .

∴ surf. A . . . K = XU · per. OM,

i.e. the inner surface = prod. XU · per. OM. [adding.

So surf. A′ . . . K′ = X′U′ · per. OA,

i.e. the outer surface = prod. X′U′ · per. OA;

and ∵ OM $\doteq$ OA and X′U′ $\doteq$ XU when the number of arcs is doubled again and again,

∴ the products XU · per. OM and X′U′ · per. OA are variables such that their difference approaches 0.

[theor. 10, lem. 2.

And ∵ the inner surface < zone AK < the outer surface,

[lem. 2.

and prod. XU · per. OM < prod. XU · per. OA

< prod. X′U′ · per. OA,

∴ there are four surfaces whereof the inner surface and its equal XU · per. OM is least; the outer surface and its equal X′U′ · per. OA is greatest; and the constant surfaces, zone AK and prod. XU · per. OA, lie between them.

If possible let zone AK ≠ prod. XU · per. OA;

then ∵ whatever the difference of these two constant surfaces, that difference is constant, and the two variables can be made to differ by less,

∴ the least of the four surfaces can be made to differ from the greatest of them by less than the two intermediate surfaces differ; which is absurd;

∴ zone AK does not differ from prod. XU · per. OA, and they are equal. Q. E. D.

(2) *The volume of the spherical sector* O–AK *is a third of the product of the zone* AK *and the radius* OA.

For $\because$ solid O–AB $= \frac{1}{3}$OM · surf. AB, [lem. 1.

solid O–BC $= \frac{1}{3}$OM · surf. BC,

. . . .

$\therefore$ solid O–A . . . K $= \frac{1}{3}$OM · surf. A . . . K, [adding.

i.e. the inner solid = prod. $\frac{1}{3}$OM and the inner surface.

So solid O–A′ . . . K′ $= \frac{1}{3}$OA · surf. A′ . . . K′,

i.e. the outer solid = prod. $\frac{1}{3}$OA and the outer surface;

and $\because$ OM $\doteq$ OA, and the inner surface $\doteq$ the outer surface,

$\therefore$ the products $\frac{1}{3}$OM by the inner surface and $\frac{1}{3}$OA by the outer surface are variables such that their difference approaches 0. [lem. 3.

And $\because$ the inner solid < sph. sect. O–AK < the outer solid, [lem. 2.

and prod. $\frac{1}{3}$OM by the inner surface < prod. $\frac{1}{3}$OA · zone AK < prod. $\frac{1}{3}$OA by the outer surface,

$\therefore$ there are four solids whereof the inner solid and its equal prod. $\frac{1}{3}$OM by the inner surface is least; the outer solid and its equal prod. $\frac{1}{3}$OA by the outer surface is greatest; and the two constant solids sph. sect. O–AK and prod. $\frac{1}{3}$OA by zone AK lie between them.

If possible let sph. sect. O–AK $\neq$ prod. $\frac{1}{3}$OA by zone AK;

then $\because$ whatever the difference of these two constants, that difference is constant, and the two variables can be made to differ by less,

$\therefore$ the least of the four solids can be made to differ from the greatest of them by less than the two intermediate solids differ; which is absurd;

$\therefore$ sph. sect. O–AK does not differ from the product $\frac{1}{3}$OA by zone AK, and they are equal. Q. E. D.